Exercise 43

Find the points on the lemniscate in Exercise 31 where the tangent is horizontal.

Solution

The equation representing the lemniscate in Exercise 31 is $2(x^2+y^2)^2 = 25(x^2-y^2)$. Differentiate both sides with respect to x.

$$\frac{d}{dx}[2(x^2+y^2)^2] = \frac{d}{dx}[25(x^2-y^2)]$$

$$2\cdot 2(x^2+y^2)\cdot \frac{d}{dx}(x^2+y^2) = 25\left[\frac{d}{dx}(x^2) - \frac{d}{dx}(y^2)\right]$$

$$2\cdot 2(x^2+y^2)\cdot(2x+2y\cdot y') = 25\left[(2x)-(2y\cdot y')\right]$$

$$8x^3+8x^2yy'+8xy^2+8y^3y' = 50x-50yy'$$

$$4x^3+4x^2yy'+4xy^2+4y^3y' = 25x-25yy'$$

Solve for y'.

$$(4x^2y+4y^3+25y)y' = 25x-4xy^2-4x^3$$

$$y' = \frac{25x-4xy^2-4x^3}{4x^2y+4y^3+25y}$$

To find where the tangent is horizontal, set $y'=0$.

$$y' = \frac{25x-4xy^2-4x^3}{4x^2y+4y^3+25y} = 0 \quad \to \quad 25x-4xy^2-4x^3 = 0 \quad \to \quad x(25-4y^2-4x^2)=0$$

This occurs when

$$x=0 \quad \text{or} \quad x^2+y^2 = \frac{25}{4}.$$

Plug each of these values into the equation for the lemniscate.

$$x=0: \quad 2(0^2+y^2)^2 = 25(0^2-y^2) \qquad \to \qquad y=0$$

$$y^2 = \frac{25}{4}-x^2: \quad 2\left[x^2+\left(\frac{25}{4}-x^2\right)\right]^2 = 25\left[x^2-\left(\frac{25}{4}-x^2\right)\right] \qquad \to \qquad x=\pm\frac{5\sqrt{3}}{4}$$

Note that having $x=0$ and $y=0$ makes y' undefined, so they're discarded. The y-values corresponding to $x=\pm5\sqrt{3}/4$ are

$$y^2 = \frac{25}{4}-\left(\pm\frac{5\sqrt{3}}{4}\right)^2 = \frac{25}{16} \quad \to \quad y=\pm\frac{5}{4}.$$

Therefore, the points on the lemniscate that have a horizontal tangent line are

$$\left(-\frac{5\sqrt{3}}{4},-\frac{5}{4}\right) \quad \text{and} \quad \left(-\frac{5\sqrt{3}}{4},\frac{5}{4}\right) \quad \text{and} \quad \left(\frac{5\sqrt{3}}{4},-\frac{5}{4}\right) \quad \text{and} \quad \left(\frac{5\sqrt{3}}{4},\frac{5}{4}\right).$$

The lemniscate is shown below with the points that have a horizontal tangent line.